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## Abstract

Triangular patches are constructed to fill in, with arbitrary order of continuity, a triangular hole within a complex of patches joining with geometric continuity. Explicit formulas are given for the special case that the hole is surrounded by rectangular patches joining with parametric continuity. Modifications and handles to control the shape of the patches are described.

## Key words

Computer Aided Geometric Design
Triangular Patches
Geometric Continuity

## 1. Introduction

Surfaces in computer aided geometric design (CAGD) are usually composed of parametric patches with rectangular or (sometimes) polygonal domain, and where certain continuity conditions between adjoining patches are imposed.

Conditions of parametric continuity, which are mainly used, are insufficient in that they do not allow for including triangular or polygonal patches into a rectangular framework, nor for an arbitrary number of patches meeting at a common vertex.

The proper condition to deal with these cases is the condition of geometric continuity, which means, in essence, the existence of a reparameterization.

The problem addressed here is that of fitting a triangular patch into a hole within a complex of patches joining with geometric continuity of arbitrary order.

Section 2 introduces the notation, formulates the problem and outlines the construction of the triangular patch.

Section 3 gives the technical part of the construction. It shows how to deal with the conditions of geometric continuity in the general case, where the patches surrounding the hole may have non-rectangular domains and join with geometric continuity.

Section 4 contains the interpolation schemes needed for the final composition of the triangular patch.

The special case where the surrounding patches are rectangular and meet with parametric continuity, is dealt with in section 5. Explicit formulas for triangular patches fitted with $GC^2$-continuity are given and handles to control their shape are described.

This article gives the details of a construction proposed in [Hahn '87]. The concepts introduced there, in particular geometric continuous patch complexes and jets, are used here without further reference.

## 2. Filling in a Triangular Hole

### The triangular hole

Assume that patches $p_i : \Delta_i \to \mathbb{R}^3$, $p_{ij} : \Delta_{ij} \to \mathbb{R}^3$ $j=1,\dots n_i$, $i=1,2,3$, form a $GC^k$-patch complex around a triangular hole with vertices $Q_i$ in the following way, see fig.1 :

(i) patches $p_i$, $i=1,2,3$ abut onto the triangular hole with adjoining edges $e_i$, parameterized as $e_i(s)$, $s \in [0,1]$, such that $p_i(e_i(0)) = Q_i$ ; $p_i(e_i(1)) = Q_{i+1}$. (Here, and subsequently, the index i will be interpreted mod 3.)

(ii) patches $p_{i-1}, p_{i1}, .., p_{in_i}, p_i$,p meet $GC^k$ with a non-convex corner at $Q_i = p_{ij}(c_{ij})$, where $c_{ij}$ is a corner of $\Delta_{ij}$. The connecting diffeomorphisms are denoted by $\phi_{i-1;i1}$, $\phi_{i1;i2}$, .., $\phi_{in_i;i}$ ,
where the (double-) index after the semicolon indicates the domain and the one before indicates the range. Accordingly the inverse diffeomorphisms are denoted by interchanging the (double-) indices.

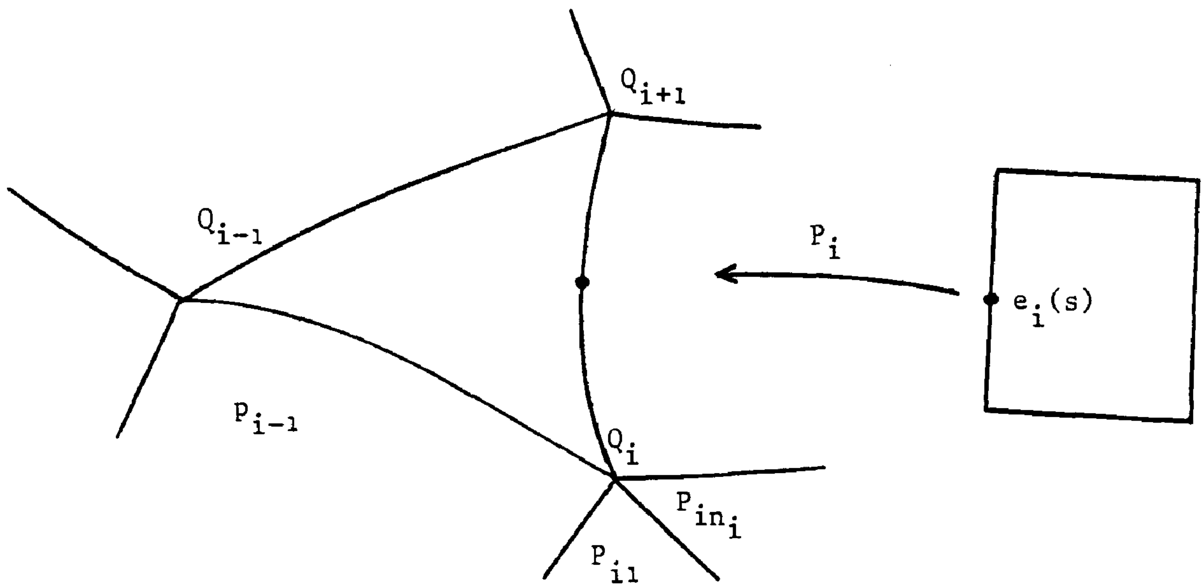


Fig.1

For instance, these diffeomorphisms transform the corners in the following way:

$$c_{ij} = \phi_{ij;i,j+1}(c_{i,j+1}), \quad c_{i,j+1} = \phi_{i,j+1;ij}(c_{ij}) .$$

The triangular domain

The hole will be filled in by a patch P, defined on an equilateral triangle $\Delta$, see fig.2, with vertices $C_i$, i=1,2,3 and edges $E_i(s) = (1-s)C_i + sC_{i+1}$ ·

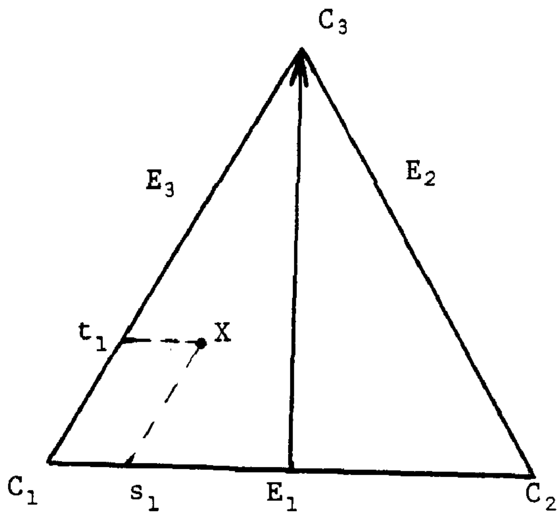


Fig.2

Let $V_i := C_{i+1} - C_i$ be tangent and $U_i := \frac{1}{2}(V_{i+1} - V_{i-1})$ be perpendicular to edge $E_i$. Let $\psi_i(X) := (s_i, t_i)$ be the affine coordinates obtained by projecting parallel to the edges adjoining at $C_i$ :

(2.1) $X = C_i + s_i V_i + t_i(-V_{i-1})$ .

Orthogonal projection gives the coordinates $(r_i, t_i)$:

(2.2) $X = C_i + r_i V_i + t_i U_i$.

Note that $(t_i, t_{i+1}, t_{i+2})$ are the barycentric coordinates of X with respect to $C_{i+2}$, $C_i$, $C_{i+1}$ (note change of order) and that

(2.3) $s_i = t_{i-1}$ , $r_i = s_i + \frac{t_i}{2}$ .

The construction

In order to have $GC^k$ joins with the abutting patches $p_i$, i=1,2,3, the patch P must satisfy the conditions

$$(2.4)\qquad j^k P_{|E_i(s)} = j^k P_{i|e_i(s)} \circ j^k \varphi_{i|E_i(s)} \quad ,$$

with connecting diffeomorphisms $\phi_i$ from $E_i(s)$ to $e_i(s)$.

Necessary for the existence of such a patch is that its jets are well-defined at the corners:

$$(2.5)\qquad j^k P_{i|e_i(0)} \circ j^k \varphi_{i|C_i} = j^k P_{i-1|e_{i-1}(1)} \circ j^k \varphi_{i-1|C_i} \quad .$$

In sec.3, connecting diffeomorphisms that satisfy these corner consistency conditions will be constructed. By interpolation, see sec.4, a patch P can be built that matches the boundary data (2.4), i.e. joins the abutting patches with $GC^k$ (To be precise, an interpolent may have points in the interior where the differential does not have maximal rank. In practice, this flaw can be cured by modifying the interpolant within the triangle, without affecting the boundary data). The tangent sector of P at $C_i$ complements the non-convex corner formed by the patches $P_{i-1}$, $P_i$, ..., $P_{in_i}$, $P_i$ at $Q_i$, without overlap; i.e. $GC^k$ is also achieved at the vertices $Q_i$.

## 3. Constructing the Connecting Diffeomorphisms

### Necessary conditions at the corners

The corner consistency condition (2.5) is equivalent to the necessary condition of theorem 7.1 of [Hahn '87] for the patches surrounding the vertex $Q_i$, which here says that

$$\text{(I)}\qquad j^k \varphi_{i|C_i} = j^k \varphi_{i,i-1|e_{i-1}}(1) \circ j^k \varphi_{i-1|C_i} \quad ,$$

where

$$j^k \phi_{i,i-1|e_{i-1}}(1) := j^k \phi_{i;i,n_i|c_{i,n_i}} \circ j^k \phi_{i,n_i;i,n_i-1|c_{i,n_i-1}} \circ \cdots \circ j^k \phi_{i,1;i-1|e_{i-1}(1)}$$

is the jet composed of the connecting diffeomorphisms between subsequent patches meeting at $Q_i$ .

Since $\phi_i$ maps edges to edges, $\phi_i(E_i(s)) = e_i(s)$, two more equations follow:

$$\text{(II)}\qquad j^k \phi_{i|C_i} \circ j^k E_{i|0} = j^k e_{i|0} \quad ,$$

$$\text{(III)}\qquad j^k \phi_{i-1|C_i} \circ j^k E_{i-1|1} = j^k e_{i-1|1} \; .$$

Computing jets at the corners

New jets $j^k\phi_i|C_i$ , $j^k\phi_{i-1}|C_i$, satisfying equations I - III, will be determined by recursively computing their derivatives in directions $V_i$ , $V_{i-1}$ :

$$\phi_i^{\,j_1,j_2}(C_i) := \partial^{\,j_1+j_2}\phi_i\Big|C_i\,(\underbrace{v_1,\dots,v_i}_{j_1},\underbrace{v_{i-1},\dots,v_{i-1}}_{j_2}) \quad ,$$

$$\phi_{i-1}^{\,j_1,j_2}(C_i) := \partial^{\,j_1+j_2}\phi_{i-1}|_{C_i}\,(V_i,\dots,V_i\;V_{i-1},\dots,V_{i-1}) \quad ,$$

$$0 \le j_1 + j_2 \le k \quad .$$

Each of the jet-equations I – III contains an equation for j-th order derivatives ($j \le k$), denoted by (I,j), (II,j), (III,j), resp. E.g. the first order equations are

(I, 1) $\quad \partial\phi_i\,|\,C_i = \partial\phi_{i,i-1}\,|\,e_{i-1}(1)\circ\partial\phi_{i-1}\,|\,C_i$ ,

(II, 1) $\quad \partial\phi_i|_{C_i}V_i = e_i'(0)$ ,

(III, 1) $\quad \partial\phi_{i-1}|_{C_i}V_{i-1} = e_{i-1}'(1)$ .

These equations determine the first order directional derivatives uniquely:

(3.1) $\quad \phi_i^{\,1,0}(C_i) = e_i'(0)$ ,

(3.2) $\quad \phi_i^{\,0-1}(C_i) = \partial\phi_{i,i-1}|e_{i-1}(1)\,(e_{i-1}'(1))$ ,

(3.3) $\quad \phi_{i-1}^{\,0,1}(C_i) = e_{i-1}'(1)$ ,

(3.4) $\quad \phi_{i-1}^{\,1,0}(C_i) = \partial\phi_{i-1,i}\Big|e_i(0)\,(e_i'(0))$ .

Here $\partial\phi_{i-1,i}|_{e_i}(0)$ is the inverse of $\partial\phi_{i,i-1}|_{e_{i-1}}(1)$ .

(Note that $\phi_i^{\,1,0}(C_i)$, $\phi_i^{\,0,1}(C_i)$ and $\phi_{i-1}^{\,1,0}(C_i)$, $\phi_{i-1}^{\,0,1}(C_i)$ are linearly independent.)

The system of equations I - III is underdetermined if k > 1. In fact, the mixed directional derivatives (the twists) of one jet can be set arbitrarily, e.g.

$$\phi_{i-1}^{\,j_1,j_2}(C_i) := 0, \quad \text{for } j_1, j_2 \ge 1 \; .$$

Proceeding by induction, assume directional derivatives

$$\phi_i^{j_1,j_2}(C_i),\quad \phi_{i-1}^{j_1,j_2}(C_i),\quad j_1 + j_2 < j,$$

are given, satisfying $(I,\ell)$, $(II,\ell)$, $(III,\ell)$ for $\ell < j$. The j-th order equations can be written as

(II, j) $\quad \phi_i^{j,0}(C_i)$ + lower order derivatives of $\phi_i = e_i^{(j)}(0)$,

(III, j) $\quad \phi_{i-1}^{0,j}(C_i)$ + lower order derivatives of $\phi_{i-1} = \ell_{i-1}^{(j)}(1)$,

and (I,j) is itself a set of j + 1 equations

$$(I; j_1, j_2)\quad \phi_1^{j_1,j_2}(C_i) = \partial\phi_{i,i-1}\Big|_{e_{i-1}(1)}\left(\phi_{i-1}^{j_1,j_2}(C_i)\right) + \text{lower order},$$

$$j_1 + j_2 = j\,.$$

The terms named 'lower order' are derived by repeated application of the chain rule and are already known by induction. Now the j-th order directional derivatives are obtained as follows:

$\phi_i^{j,0}(C_i)$ and $\phi_{i-1}^{0,j}(C_i)$ are given by (II,j), (III,j) resp.,

$\phi_{i-1}^{j_1,j_2}(C_i)$, $j_1, j_2 \geq 1$ was already chosen above,

$\phi_i^{j_1,j_2}(C_i)$, $j_2 \leq j$ is given by $(I; j_1\, j_2)$, and

$\phi_{i-1}^{j,0}(C_i)$ is given by (I,j,0), since $\partial\phi_{i,i-1}|_{e_{i-1}(1)}$ is invertible.

The directional derivatives $\phi_i^{j_1,j_2}(C_i)$, $\phi_{i-1}^{j_1,j_2}(C_i)$, $J_1 + J_2 \leq k$, describe jets $j^k\phi_i\big|_{C_i}$ and $j^k\phi_{i-1}\big|_{C_i}$ that satisfy conditions I - III. Moreover $\partial\phi_i\big|_{C_i}$ and $\partial\phi_{i-1}\big|_{C_i}$ have maximal rank. By corner consistency (2.5), $\partial p_i\big|_{e_i(0)}\cdot\partial\phi_i\big|_{C_i}(-V_{i-1})$ is the tangent vector of the curve $p_{i-1}\circ e_{i-1}$ and points to the side of $p_i \circ e_i$ opposite to the tangent sector of $p_i$ at $e_i(0)$, because the corner at $Q_i$ is non-convex. Therefore $\partial\phi_i\big|_{C_i}$ maps the inward pointing vector $-V_{i-1}$ of $\Delta$ to a vector pointing to the exterior side of edge $e_i$. Similarly, $\partial\phi_{i-1}\big|_{C_i}$ maps the inward pointing vector $V_i$ to an outward pointing vector of $\Delta_{i-1}$.

Diffeomorphisms along the edges

Now connecting diffeomorphisms $\phi_i$ can be constructed whose jets at the corners $C_i$, $C_{i+1}$ are $j^k\phi_i\big|C_i$ and $j^k\phi_i\big|C_{i+1}$, as computed above.

The jets can be represented by vectors

$$W_{i,0}^{j_1,j_2} := \partial^{j_1+j_2}\phi_i\big|_{C_i}(\underbrace{U_i,\ldots,U_i}_{j_1},\underbrace{V_i,\ldots,V_i}_{j_2})$$

$$W_{i,1}^{j_1,j_2} := \partial^{j_1+j_2}\phi_i\big|_{C_{i+1}}(U_i,\ldots,U_i,V_i,\ldots,V_i),\ \le\ j_1+j_2\le k\ .$$

Let $W_i^j(s)$, $1 \le j \le k$ be vector fields along edge $e_i(s)$ such that $W_i^j$ is $C^{k-j}$,

$$\frac{d^\ell}{ds^\ell} W_i^j(0) = W_{i,0}^{j,\ell}\ ,$$

$$\frac{d^\ell}{ds^\ell} W_i^j(1) = W_{i,1}^{j,\ell},\quad 0 \le \ell \le k-j\ ,$$

and such that in addition $W_i^1(s)$ is transversal to $e_i(s)$ and outward pointing. The vector fields $W_i^j(s)$, $j=2,..k$, can be obtained by e.g. Hermite interpolation. For the vector field $W_i^1(s)$ it may be necessary to modify an interpolant to ensure transversality; this is always possible since $W_{i,0}^{1,0}$ and $W_{i,1}^{1,0}$ are both outward pointing.

By lemma 3.2 of [Hahn '87], there exist $C^k$-connecting diffeomorphisms $\phi_i$ from $E_i$ to $e_i$ with

$$\partial^j\phi_i\ E_i(s)\,(U_i,\ldots,U_i) = W_i^j(s)\ .$$

At the corners, these $\phi_i$ have the jets computed above and meet the corner consistency condition (2.5).

## 4. Interpolation within a Triangle

With the connecting diffeomorphisms $\phi_i$, boundary data for the triangle,

(4.1) $\quad \partial^j(p_i \circ \phi_i)\big|_{E_i(s)},\quad j=0,\ldots,k,$

which are consistent at the corners $C_i = E_i(0) = E_{i-1}(1)$, can be computed.

Now an interpolant with these boundary data needs to be constructed.

So far, it has been assumed that the abutting patches $p_i$ are just $C^k$, which means that the derivative $\partial^j p_i$ may be only $C^{k-j}$. In this case quite sophisticated interpolation techniques, as in lemma 3.2 of [Hahn '87] are needed to ensure that the interpolant is $C^k$.

In practice, however, the patches will have much higher order of continuity. For this case explicit interpolation formulas will be given.

A Boolean sum type interpolant

Within the affine coordinates $\psi_i(X) = (s_i, t_i)$ (2.1) define a Boolean sum Taylor interpolant

$$(4.2) \qquad p_i(s,t) := \sum_{j=0}^{k} \frac{t^j}{j!} \partial_{0,j} (p_i \circ \phi_i \circ \psi_i^{-1})(s,0) + \sum_{j=0}^{k} \frac{s^j}{j!} \partial_{j,0} (p_{i-1} \circ \phi_{i-1} \circ \psi_i^{-1})(0,t) - \sum_{j_1,j_2=0}^{k} \frac{s^{j_1} t^{j_2}}{j_1! j_2!} \partial_{j_1,j_2} (p_i \circ \phi_i \circ \psi_i^{-1})(0,0) \ ,$$

$\left(\partial_{j_1,j_2} := \dfrac{\partial^{j_1+j_2}}{\partial s^{j_1} \partial t^{j_2}} \text{ denotes partial differentiation}\right)$. Then

$$(4.3) \quad P_i \circ \psi_i(X) = P_i(s_i, t_i)$$

is a two-side interpolant on the triangle.

$P_i \circ \psi_i$ is $C^k$ and matches the boundary data (4.1) along edges $E_i$ and $E_{i-1}$, provided the cross-boundary derivatives $\partial_{0,j}(P_i \circ \phi_i \circ \psi_i^{-1})(s,0)$ and $\partial_{j,0}(P_{i-1} \circ \phi_{i-1} \circ \psi_i^{-1})(0,t)$ are $C^k$-functions of s, t resp. and the twists at the corner coincide

$$(4.4) \quad \partial_{j_1,j_2}(P_i \circ \phi_i \circ \psi_i^{-1})(0,0) = \partial_{j_1,j_2}(P_{i-1} \circ \phi_{i-1} \circ \psi_i^{-1})(0,0) \ .$$

The triangular patch is a convex combination of these two-side interpolants:

$$(4.5) \quad P(X) := \sum_{i=1}^{3} W_i(X) \, P_i(s_i, t_i) \ ,$$

where $w_i$ are weight functions that sum up to unity and vanish up to order k along edge $E_{i+1}$, e.g.

$$(4.6)\quad w_i(X) = \frac{t_{i+1}^{k+1}}{\sum_{j=1}^{3} t_j^{k+1}}$$

A one-side interpolant

An alternative interpolant is

$$(4.7)\qquad P_i(r,t) := \sum_{j=0}^{k} \frac{t^j}{j!}\, \partial^j (P_i \circ \phi_i) \Big|\, E_i(r)\, (U_{i,..,}U_i)\, .$$

$P_i(r_i,t_i)$ is $C^k$ on the triangle and matches the boundary data (4.1) along edge $E_i$, provided the cross-boundary derivatives $\partial^j(P_i \circ \phi_i)_{E_i}(r)(U_i,\dots,U_i)$ are $C^k$-functions of r . Convex combination of these interpolants gives an alternative patch

$$(4.7)\qquad P(X) := \sum_{i=1}^{3} \frac{t_{i-1}^{k+1}\, t_{i+1}^{k+1}}{\sum_{j=1}^{3} t_{j-1}^{k+2}\, t_{j+1}^{k+1}} P_i(r_i, t_i)\ .$$

Although the weight functions are singular at the corners, it can be shown that the patch P is indeed $C^k$.

5. Application; A Triangular Patch within a Complex of Rectangular Patches Joining with Parametric Continuity

Assumptions

Assume that patches $p_i :\ [0,1]^2 \to \mathbb{R}^3$ , i =1, 2, 2, surround a triangular hole, see fig. 3, such that the composed map

$$(5.1)\quad (u,v) \to \begin{cases} p_{i-1}(u,v), & \text{for}\,(u,v) \in [0,1]^2 \\ p_i(v-1,-u), & \text{for}\,(u,v) \in [-1,0]\times[1,2] \end{cases}$$

is (parametric) $C^k$-continuous for i=1,2,3 (mod 3).

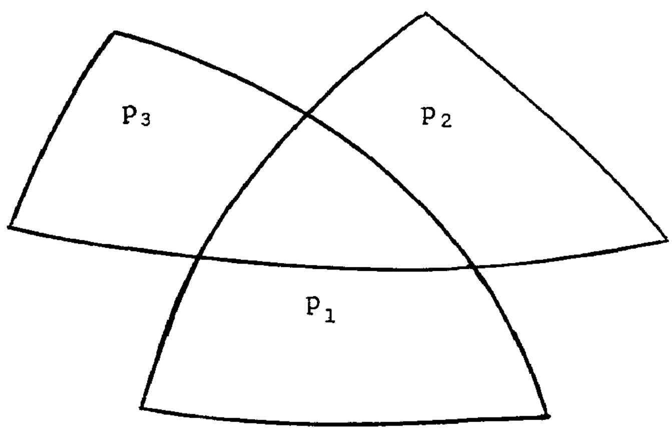


Fig.3

The connecting diffeomorphisms

$c^k$ -continuity of the composed map (5.1) means that $\phi_{i,i-1}$ , the composed connecting diffeomorphism from patch $p_{i-1}$ to $p_i$, is the rotation by $-\frac{\pi}{2}$ around $\left(-\frac{1}{2}, \frac{1}{2}\right)$. According to (3.1-3.4), the first order derivatives of the connecting diffeomorphisms $\phi_i$, $\phi_{i-1}$ at the corner $C_i$ are computed as

$$\begin{aligned}
&\partial\phi_{i|C_i} V_i = (0,1) ,\\
&\partial\phi_{i|C_i} V_{i-1} = (1,0) ,\\
&\partial\phi_{i-1|C_i} V_i = (1,0) ,\\
&\partial\phi_{i-1|C_i} V_i = (-1,0) ,
\end{aligned}$$

and the higher order derivatives can be set to zero, since those of $\phi_{i,i-1}$ vanish, cf example 7.2 of [Hahn '87].

Now connecting diffeomorphisms can be constructed more easily than in sec.3 as a blend of the affine linear maps $\phi_i(C_i) + \partial\phi_{i|C_i}(X - C_i)$ and $\phi_i(C_{i+1}) + \partial\phi_{i|C_{i+1}}(X - C_{i+1})$. In terms of the coordinates $(r_i, t_i)$ (2.2), set

$$\phi_i(X) := \left(-t_i,\ r_i + \frac{t_i}{2}(\beta(r_i) - \alpha(r_i))\right) , \tag{5.2}$$

where $\alpha, \beta$ are $C^k$-functions such that

$$\begin{aligned}
&\alpha(s),\ \beta(s) \geq 0 ,\\
&\alpha(0) = \beta(1) = 1,\ \ \alpha(1) = \beta(0) = 0
\end{aligned}$$

and all derivatives up to order k at 0 and 1 vanish.

The one-side interpolant

The derivatives needed for the one-side interpolant (4.7) are

$$\begin{aligned}
\partial^j(p_i \circ \phi_i)_{|E_I(r)}(U_i, \ldots, U_i) &= \frac{d^j}{dt^j}\, p_i \circ \phi_i(E_i(r) + tU_i)_{|t=0} \\
&= \frac{d^j}{dt^j}\, p_i\left(-t,\ r + \frac{t}{2}(\beta(r) - \alpha(r))\right)_{|t=0} \\
&= \sum_{\ell=0}^{j} \binom{j}{\ell}(-1)^{\ell} \left(\frac{\beta(r) - \alpha(r)}{2}\right)^{j-\ell} \partial_{\ell, j-\ell}\, p_i(0,r) \ .
\end{aligned} \tag{5.3}$$

In the case k = 2, the interpolant is explicitly:

(5.4) $p_i(r,t) = p_i(0,r)$

$$+\ t\left(-\partial_{10}\, p_i(0,r) + \frac{\beta(r)-\alpha(r)}{2}\,\partial_{01}\, p_i(0,r)\right)$$

$$+\ \frac{t^2}{2}\left(\partial_{20}\, p_i(0,r) - (\beta(r)-\alpha(r))\,\partial_{11}\, p_i(0,r) + \left(\frac{\beta(r)-\alpha(r)}{2}\right)^2 \partial_{02}\, p_i(0,r)\right),$$

where $\alpha,\beta$ are $C^2$ -functions such that

$$\alpha(r), \beta(r) \geq 0\ ,$$
$$\alpha(0) = \beta(1) = 1, \alpha(1) = \beta(0) = 0\ ,$$

and whose first and second derivative vanish at 0 and 1 ;

(5.5) $$P(X) = \sum_{i=1}^{3} \frac{t_{i-1}^3\ t_{i+1}^3}{\sum_{j=1}^{3} t_{j-1}\ t_{j+1}}\ P_i(r_i, t_i)\ .$$

The interpolant is $C^2$ provided the partial derivatives $\partial_{j_1, j_2}\, p_i(0,r)$, $j_1 + j_2 \leq 2$, are $C^2$-functions of r .

<u>The Boolean sum type interpolant</u>

The Boolean sum type interpolant (4.2) can be employed if stronger continuity conditions are imposed.

Since

(5.6) $$\phi_i \circ \psi_i^{-1}(s,t) = (-t,\ s + t\beta(s + \tfrac{t}{2}))$$

and

(5.7) $$\phi_{i-1} \circ \psi_i^{-1}(s,t) = (-s,\ 1 - t - s\alpha(1 - t - \tfrac{s}{2}))\ ,$$

the cross-boundary derivatives needed are:

(5.8) $$\partial_{0,j}(p_i \circ \phi_i \circ \psi_i^{-1})(s,0) = \frac{d^j}{dt^j}\, p_i(-t,\ s + t\beta\ (s + \tfrac{s}{2}))_{\,|\, t=0}\ ,$$

and

(5.9) $$\partial_{j,0}(p_{i-1} \circ \phi_{i-1} \circ \psi_i^{-1})(0,t) = \frac{d^j}{ds^j}\, p_{i-1}\ (-s,\ 1 - t - s\alpha\ (1 - t - \tfrac{s}{2}))_{\,|\, s=0}$$

These contain already derivatives of the functions $\alpha,\beta$.

It is therefore assumed that $\alpha$ and $\beta$ are (2k-1) - times continuously differentiable and that their derivatives vanish up to order 2k-1 at 0 and 1 .

Then the mixed derivatives at the corner $(s,t) = (0,0)$ can be obtained by observing that in (5.6), (5,7) the terms $t\beta(s+\frac{t}{2})$ and $s\alpha\ (1-t-\frac{s}{2})$ vanish up to order 2k at (0,0):

(5. 10) $$\partial_{j1,j2}(p_i \circ \phi_i \circ \psi_i^{-1})(0,0) = (-1)^{j2}\partial_{j2,j1}p_i(0,0) ,$$

(5.11) $$\partial_{j1,j2}(p_{i-1} \circ \phi_{i-1} \circ \psi_i^{-1})(0,0) = (-1)^{j1+j2}\partial_{j1,j2}\ p_{i-1}\ (0,1) .$$

These coincide, if

(5.12) $$\partial_{j1,j2}\ p_{i-1}(0,1) = (-1)^{j_2}\ \partial_{j2,j1}\ p_i(0,0), \quad \text{for } j_1, j_2 \le k ,$$

which means that the composed map (5.1) is $C^{k,k}$ at (0,1). If, in addition, the abutting patches $p_i$ are $C^2$k, then the interpolant (4.2) is $C^k$ and matches the boundary data.

In the case k = 2, this interpolant is explicitly:

(5.13)
$$\begin{aligned}
p_i(s,t) = {} & p_i(0,s) \\
& + t(-\partial_{10}p_i(0,s) + \beta(s)\partial_{01}p_i(0,s)) \\
& + \frac{t^2}{2}(\partial_{20}p_i(0,s) - 2\beta(s)\partial_{11}p_i(0,s) \\
& \qquad + \beta^2(s)\ \partial_{02}p_i(0,s) \\
& \qquad + \beta'(s)\ \partial_{01}p_i(0,s)) \\
& + p_{i-1}(0,1-t) \\
& + s(-\partial_{10}p_{i\cdot 1}(0,1-t) - \alpha(1-t)\partial_{01}p_{i-1}(0,1-t)) \\
& + \frac{s^2}{2}(\partial_{20}p_{i-1}(0,1-t) + 2\alpha(1-t)\partial_{11}p_{i-1}(0,1-t) \\
& \qquad + \alpha^2(1-t)\partial_{02}p_{i-1}(0,1-t) \\
& \qquad + \alpha'(1-t)\partial_{01}p_{i-1}(0,1-t) \\
& - p_i(0,0) \\
& - s\partial_{01}p_i(0,0) + t\partial_{10}p_i(0,0) \\
& - \frac{s^2}{2}\partial_{02}p_i(0,0) + st\partial_{11}p_i(0,0) - \frac{t^2}{2}\partial_{20}p_i(0,0) \\
& + \frac{s^2t^2}{2}\partial_{12}p_i(0,0) - \frac{st^2}{2}\partial_{21}p_i(0,0) \\
& - \frac{s^2t^2}{4}\partial_{22}p_i(0,0)
\end{aligned}$$

Here, $\alpha, \beta$ are $C^3$-functions such that

$$\alpha(s), \beta(s) \ge 0 ,$$
$$\alpha(0) = \beta(1) = 1, \alpha(1) = \beta(0) = 0,$$

and whose derivatives up to order 3 at 0 and 1 vanish. The patches $p_i$. must be 4-times continuously differentiable and (5.12) must hold with k = 2.

The triangular patch is

$$(5.14) \qquad P(X) := \sum_{i=1}^{3} \frac{t_{i+1}^3}{\sum_{j=1}^{3} t_j^3} p_i(s_i, t_i) .$$

Modifications and adaptions

The modifications described in [Gregory and Hahn '87] for a polygonal patch can be employed.

In (5.13) the terms

$$\frac{t^2}{2} \beta'(s) \partial_{01} p_i(0,s) \text{ and } \frac{s^2}{2} \alpha'(1-t) \partial_{01} P_{i-1}(0,1-t)$$

contribute, along the sides, only to the tangent part of the second derivative. They may be omitted, without affecting geometric continuity, cf [Gregory and Hahn '86]. Moreover, in this case, the blending functions $\alpha, \beta$ may be chosen to be only twice continuously differentiable.

The effect of the abutting patches can be varied individually by changing the connecting diffeomorphisms. In (5.2) $\alpha, \beta$ can be substituted by $\alpha_i, \beta_i$ and these functions can be adapted appropriately.

The interior shape of the triangular patch can be controlled by adding a function which vanishes up to its k-th derivatives along the edges, e.g. $(t_1 t_2 t_3)^{k+1} Q(X)$.

References

Gregory, J.A. and Hahn, J.M. (1986), Geometric continuity and convex combination patches, to appear in Computer Aided Geometric Design.

Gregory, J.A. and Hahn, J.M. (1987), Polygonal patches of high order continuity, Technical Report TR/01/87, Dept. Maths and Stats, Brunei University.

Hahn, J.M. (1987), Geometric continuous patch complexes, Technical Report TR/05/87, Dept. Maths, and Stats, Brunel University.